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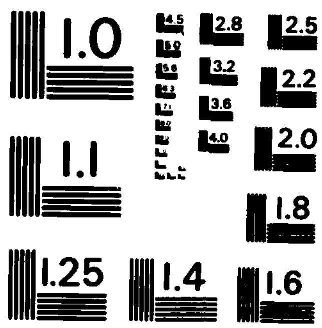

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ROBUST PREDICTION AND INTERPOLATION FOR
VECTOR STATIONARY PROCESSES-PART 3

by

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ROBUST PREDICTION AND INTERPOLATION
FOR VECTOR STATIONARY PROCESSES

by

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Abstract

Robust multivariate prediction and interpolation problems for statistically contaminated vector valued second order stationary processes are considered. The statistical contamination is modeled by requiring that the spectra of the processes lie within certain nonparametric classes. Both prediction and interpolation are then formalized as games whose saddle point solutions are sought. Finally, such solutions are found and analyzed, for two specific multivariate spectral classes.

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1. Introduction

The prediction and interpolation problems for stationary processes have received considerable attention for a number of years. The bulk of the work concentrates around scalar processes and the parametric model. The assumption there is that the measure generating the stochastic process is well known. The initial significant results on prediction and interpolation for the parametric model were given by Wiener (1949) and Kolmogorov (1941).

Strictly speaking, the term prediction refers to the extraction of a datum from the process, when a number of past process data have been observed noiselessly. The term interpolation refers to the same extraction, when past as well as future noiseless process data are available. The two terms are extended sometimes to include noisy observation data. Some results on those extended problems, and for the parametric model, can be found in the papers by Snyders (1973) and Viterbi (1965). We point out here that the majority of studies on the extended problems consider asymptotic and linear prediction and interpolation operations.

The last few years considerable attention has been given to the robust extended prediction problem. Some attention has also been given to the robust nonextended interpolation problem. The robust model is nonparametric, and the assumption is that the measure that generates the stationary process is not well known. The existing work on robust extended prediction and interpolation concentrates around scalar stationary processes, linear asymptotic prediction and interpolation operations, and noisy observation data. Representative results here include robust Wiener and Kalman filtering for scalar stationary processes (Masreliez et al (1977), Kassam et al (1977), Martin et al (1976), Cimini et al (1980), Poor (1980)). Related work on time series outliers can be found in Martin et al (1977). Hosoya (1978) considers the robust nonextended prediction problem, for linear contaminated scalar stationary processes. The robust solution is found there within the class of asymptotic linear prediction operations. A game theoretic formulation on the measures of the stochastic processes is presented by Papantoni-Kazakos (1984), for the robust extended prediction problem. Chen et al (1981, 1982) consider robust multidimensional matched filtering, for classes with identical eigenvectors. Regarding the robust nonextended interpolation problem, for scalar processes, the interested reader may look into the works by Taniguchi (1981) and Kassam (1982).

The prediction problem for vector processes is considerably more involved than that for scalar processes. The difficulty is mainly due to the cross correlations among the component processes, which have a direct impact on the complexity of the correlation matrix, and the spectral distribution matrix of the vector process. Important questions regarding the structure of a vector process such as rank, regularity, and non-determinacy are treated by Wiener et al (1957), (1958), Helson et al (1958), Hannan (1970), and Zasuhin (1941).

In the present paper, we consider the robust nonextended prediction and interpolation problems for vector stationary processes. Vector processes have not been treated in this case (some limited consideration for the interpolation problem can be found in Taniguchi (1981)), and they present interesting peculiarities both theoretical and practical. We will generally assume that the spectrum of the vector process, which is represented by the spectral distribution matrix, belongs to a class of spectra, and we will formulate the prediction and interpolation problems as games with saddle point solutions. Then, we will find those solutions for two specific spectral classes. One of the classes represents linear contamination of a nominal spectral matrix. The other class includes the set of all spectral matrices with fixed energy on prespecified frequency quantiles.

The organization of the paper is as follows. In section 2, we give a background on the multivariate prediction and interpolation problems. In section 3, we define the spectral classes under consideration, and we formalize the prediction and interpolation games. In sections 4 and 5, we find the saddle point solutions for the prediction and interpolation games respectively. Finally, in section 6 we present some conclusions and a brief discussion.

2. Preliminaries

The original linear prediction problem, for an n-variable, discrete-time, second order stationary process, $\{\underline{x}_k,\ k \in Z\}$, is equivalent to searching for a matrix trigonometric polynomial $g_p(e^{j\omega})$ of the form

$$g_p(e^{j\omega}) = I + \sum_{i=1}^{N} A_i\, e^{j\omega i} \tag{1}$$

which makes the mean square error of prediction

$$e_p(F(\omega),\ g_p(e^{j\omega})) = \frac{1}{2\pi}\, \mathrm{tr} \int_{-\pi}^{\pi} g_p(e^{j\omega}) d\, F(\omega)\ g_p^{*T}(e^{j\omega}) \tag{2}$$

as small as possible. In (1), N runs over all the positive integers, A_1, A_2 ... are any nxn complex matrices, and I is the nxn identity matrix. In (2), $F(\omega)$ denotes the spectral distribution matrix of the process, which defines a positive definite Hermitian finite matrix-valued measure, on the Borel field of the measurable space $[-\pi,\pi]$, with $\frac{1}{2\pi} \int_{-\pi}^{\pi} d\, F(\omega) = R_o = E\, \{\underline{x}_k\ \underline{x}_k^{*T}\}$. The symbols *, T, and tr stand for conjugate, transpose, and trace, respectively.

Let S_p be the convex set of polynomials of the form (1). We define the space $L_2(dF(\omega))$ of all nxn matrix-valued functions $A(\omega)$ on $[-\pi,\pi]$, for which (3) below is true.

$$\mathrm{tr} \int_{-\pi}^{\pi} A(\omega) d\, F(\omega)\ A^{*T}(\omega) < \infty \tag{3}$$

Then, $S_p \subset L_2(d\, F(\omega))$. Considering any two elements, $A_1(\omega)$ and $A_2(\omega)$, of $L_2(d\, F(\omega))$ as equivalent, iff $\mathrm{tr} \int_{-\pi}^{\pi} (A_1(\omega)-A_2(\omega)) d\, F(\omega)\ (A_1(\omega)-A_2(\omega))^{*T} = 0$, then, $L_2(d\, F(\omega))$ is made into a Hilbert space (Hannan (1970)), with inner product and norm defined respectively, as follows.

$$(A_1(\omega),\ A_2(\omega))_{d\ F(\omega)} = \mathrm{tr} \int_{-\pi}^{\pi} A_1(\omega)\ d\ F(\omega)\ A_2^{*T}(\omega)$$

$$||A_1||_{d\ F(\omega)} = (A_1(\omega),\ A_1(\omega))^{1/2}$$

Under the new notation, (2) can be rewritten as follows.

$$e_p(F(\omega),\ g_p(e^{j\omega})) = (2\pi)^{-1} ||g_p(e^{j\omega}))||^2_{d\ F(\omega)} \tag{4}$$

Now, let $\bar{S}_p$ (d F(ω)) be the closure of S_p in L_2(d F(ω)). Since $\bar{S}_p$(d F(ω)) is a closed and convex set in the Hilbert space L_2(d F(ω)), it contains a unique element of smallest norm (unique in the equivalence sense defined above). It follows that the infimum in (2) with respect to $g_p(e^{j\omega})$ is attained in $\bar{S}_p$(d F(ω)).

For reasons that will be explained below, we are going to consider a more general prediction problem, by enlarging the set S_p, to contain all matrix trigonometric polynomials $g_p^o(e^{j\omega})$ of the form,

$$g_p^o(e^{j\omega}) = A_o + \sum_{i=1}^{N} A_i\ e^{j\omega i} \tag{5}$$

As before, N runs over all the positive integers, and A_1, A_2 ... are any complex nxn matrices. A_o, however, can now be any nxn complex matrix, whose determinant is constrained to be equal to 1. Let S_p^o be the set of all polynomials of the form (5). The convex hull S_p^{oc} of S_p^o will contain all polynomials of the form $B_o + \sum_{i=1}^{N} B_i\ e^{j\omega i}$, with det $(B_o) \geq 1$. This follows from the concavity of the function det (·); namely, det $(\lambda A+(1-\lambda)B) \geq (\det(A))^{\lambda}(\det(B))^{1-\lambda}$; $0 \leq \lambda \leq 1$ (Bellman (1970)). If we take the closure $\overline{S_p^{oc}}$(d F(ω) of S_p^{oc} in L_2(d F(ω)), we can define the new prediction problem as follows.

$$\min_{g_p^o(e^{j\omega}) \in \overline{S_p^{oc}}(d\ F(\omega))} e_p(F(\omega), g_p^o(e^{j\omega})) = (2\pi)^{-1} \min_{g_p^o(e^{j\omega}) \in \overline{S_p^{oc}}(d\ F(\omega))} ||g_p^o(e^{j\omega})||_{d\ F(\omega)} \tag{6}$$

As a result of the derivation of the optimal predictor in Helson et al (1958), the minimum in (6) assumes a closed form expression. In particular,

$$\min_{g_p^o(e^{j\omega})\epsilon \overline{S_p^{oc}}(d\ F(\omega))} e_p(F(\omega),g_p^o(e^{j\omega})) = n\ \exp[(2\pi n)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ell og f(\omega)d\omega] \tag{7}$$

where $f(\omega)d\omega$ is the absolutely continuous part of the measure $d\ F(\omega)$, with respect to the Lebesque measure in $[-\pi,\pi]$, and where $f(\omega)$ is the spectral density matrix of the process, which is Hermitian, nonnegative definite a.e. $(d\omega)$, and integrable in $[-\pi,\pi]$. If the scalar function $\mathrm{tr}\ell og f(\omega)$ in (7) is not integrable (since $\int_{-\pi}^{\pi} \mathrm{tr}\ell og f(\omega)d\omega$ is bounded from above, as can be verified by using Jensen's inequality, this can only happen if $\int_{-\pi}^{\pi} \mathrm{tr}\ell og\ f(\omega)d\omega = -\infty$), the right hand side of (7) is interpreted as zero. An element $g_p^{o'}(\omega)$ in $\overline{S_p^{oc}}(dF(\omega))$, that attains the minimum in (6) is such that,

$$g_p^{o'}(\omega)(g_p^{o'}(\omega))^{*T} = \exp[(2\pi n)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ell og f(\omega)d\omega\ I-\ell og f(\omega)] \tag{8}$$

for points $\omega\epsilon[-\pi,\pi]$, such that $f(\omega)$ exists (i.e. a.e. $(d\omega)$), and $g_p^{o'}(\omega) = 0$ for an at most countable subset of $[-\pi,\pi]$. The latter is simply a manifestation of the fact that the singular part of $d\ F(\omega)$ contributes nothing to the minimum in (7), and it corresponds to a purely deterministic process. If $g_p^{o'}(\omega)$ exists, it is then proven by Helson et al (1958) that $g_p^{o'}(\omega)\epsilon\ \overline{S_p^o}\ (d\ F(\omega))$; i.e. the determinant of its leading Fourier coefficient is equal to 1, or equivalently the minimum in (6) for $g_p^o(\omega)\epsilon\ \overline{S_p^o}\ (d\ F(\omega))$ exists, although $\overline{S_p^o}\ (d\ F(\omega))$ is not convex, and it is attained at $g_p^{o'}(\omega)$.

Consideration of the prediction problem in $\overline{S_p^{oc}}\ (d\ F(\omega))$, or equivalently in $\overline{S_p^o}\ (d\ F(\omega))$, has the remarkable advantage of a simple closed form expression for the minimum error given by (7), which is a direct generalization of Szegö's, formula (Grenander et al (1958)), for scalar processes.

The linear interpolation problem for an n-variable discrete-time second order stationary process $\{\underline{x}_k,\ k \in Z\}$ is less difficult, because the constraint set of the associated minimization problem is larger than $\overline{S_p}$ (d $F(\omega)$), since no causality requirement exists. We will denote by S_i the convex set of all trigonometric polynomials of the form,

$$g_i(e^{j\omega}) = I + \sum_{\substack{i=-N \\ i \neq 0}}^{N} A_i\ e^{j\omega i} \tag{9}$$

where N runs over all positive integers, and $\{A_i,\ i \neq 0\}$ run over all complex nxn matrices. The error that has to be minimized is similar to that in (2), and it is rewritten here for completeness.

$$e_i(F(\omega), g_i(e^{j\omega})) = (2\pi)^{-1}\ \mathrm{tr} \int_{-\pi}^{\pi} g_i(e^{j\omega}) d\ F(\omega) g_i^{*T}(e^{j\omega}) = (2\pi)^{-1} \| g_i(e^{j\omega}) \|^2_{dF(\omega)} \tag{10}$$

Taking the closure $\overline{S_i}$ (d $F(\omega)$) of S_i in L_2(d $F(\omega)$), we can define the interpolation problem as follows.

$$\min_{g_i(\omega) \in \overline{S_i}(d\ F(\omega))} e_i(F(\omega), g_i(e^{j\omega})) = (2\pi)^{-1} \min_{g_i(\omega)\ \overline{S_i}(d\ F(\omega))} \| g_i(e^{j\omega})) \|^2_{dF(\omega)} \tag{11}$$

As derived in Hannan (1970), the minimum in (11) is given by

$$\min_{g_i(\omega) \in \overline{S_i}(d\ F(\omega))} e_i(F(\omega), g_i(\omega)) = 2\pi\ \mathrm{tr}\ [(\int_{-\pi}^{\pi} f^{-1}(\omega) d\omega)^{-1}] \tag{12}$$

and it is attained for some $g_i^{\prime}(\omega) \in \overline{S_i}$ (d $F(\omega)$), such that

$$g_i^{\prime}(\omega) = 2\pi\ (\int_{-\pi}^{\pi} f^{-1}(\omega) d\omega)^{-1}\ f^{-1}(\omega)\ ;\ \text{a.e.}\ (d\omega) \tag{13}$$

$g_i(\omega) = 0$ for an at most countable subset of $[-\pi,\pi]$, where the singularities of the spectrum are located. In (13), $f^{-1}(\omega)$ is the Penrose-Moore generalized inverse of $f(\omega)$ and it is integrable for full rank processes.

3. The Robust Formalization

We now look at the above problems from a different point of view. We assume that the spectral structure of the process is only vaguely or incompletely specified. This corresponds to a more realistic situation, since the procedures for obtaining the spectrum of a process always involve errors. This applies even more to vector processes, where the increased complexity results in larger errors. With the above in mind, it is clear that new formalizations of the problems considered in section 2 are needed. Such a formalization is given below, where the spectral distribution matrix of the process is assumed to be a member of a whole class of spectral distribution matrices. For the purpose of this work we are going to consider two different types of spectral classes, denoted by $\mathcal{F}_L$ and $\mathcal{F}_Q$, which are defined as follows:

(a) $\mathcal{F}_L = \{F(\omega)/F(\omega) = (1-\varepsilon)F_o(\omega) + \varepsilon\, H(\omega),\ \omega\varepsilon[-\pi,\pi], \varepsilon$ fixed and such that, $0 < \varepsilon < 1.$

$F_o(\omega)$: well defined fixed nominal spectral distribution matrix

$H(\omega)$: arbitrary spectral distribution matrix satisfying

$$(2\pi)^{-1}\ \mathrm{tr} \int_{-\pi}^{\pi} d\, H(\omega) = (2\pi)^{-1}\ \mathrm{tr} \int_{-\pi}^{\pi} d\, F_o(\omega) = W > 0,\ W \text{ fixed}\}$$

(b) $\mathcal{F}_Q = \{F(\omega)/(2\pi)^{-1}\mathrm{tr}\int_{D_i} d\, F(\omega) = c_i > 0,\ i=1,\ldots k,\ c_1,\ldots c_k$ fixed

$D_1,\ldots D_k$ fixed Lebesque measurable subsets of $[-\pi,\pi]$ with positive measure each and

$$D_i \cap D_j = \emptyset; \forall\ i \neq j,\ \bigcup_{i=1}^{k} D_i = [-\pi,\pi]\}$$

$\mathcal{F}_L$ is called the ε-contaminated class, or the gross error model, and it corresponds to the situation where the nominal process $F_o(\omega)$ occurs only with probability $1-\varepsilon < 1$, while with probability ε, any other process, with the specified energy constraints, may occur. $\mathcal{F}_Q$ is called the p-point class, and it contains all the spectra, whose energy is specified by a positive number, on a finite collection of mutually exclusive and exhaustive measurable subsets of $[-\pi,\pi]$, with positive Lebesque measure.

In pursuing a robust formalization, for prediction and interpolation, it will be necessary to restrict the classes $\overline{S_p^o}$ (d F(ω)), $\overline{S_i}$ (d F(ω)) defined in section 2, for the simple reason that instead of a single F(ω), a whole class of spectral matrices is considered. In particular, we will consider the following classes of predictors and interpolators.

$$S_{pL} = \bigcap_{F(\omega)\varepsilon \mathcal{F}_L} \overline{S_p^o}\ (d\ F(\omega)) \qquad S_{iL} = \bigcap_{F(\omega)\varepsilon \mathcal{F}_L} \overline{S_i}\ \ (d\ F(\omega))$$

$$S_{pQ} = \bigcap_{F(\omega)\varepsilon \mathcal{F}_Q} \overline{S_p^o}\ (d\ F(\omega)) \qquad S_{iQ} = \bigcap_{F(\omega)\varepsilon \mathcal{F}_Q} \overline{S_i}\ \ (d\ F(\omega)) \tag{14}$$

We are now in a position to formalize the following games:
Find pairs $(F_T^e(\omega), g_{RT}^e(\omega))\varepsilon \mathcal{F}_T x S_{RT}$ $(T = L,Q.\ R = p,i)$ such that

$$e_R(F_T(\omega), g_{RT}^e(\omega)) \le e_R(F_T^e(\omega), g_{RT}^e(\omega)) \le e_R(F_T^e(\omega), g_{RT}(\omega))$$

$$;\ \forall\ F_T(\omega)\varepsilon \mathcal{F}_T,\ \forall\ g_{RT}(\omega)\varepsilon\ S_{RT} \tag{15}$$

If the pairs with the superscript e exist, we call them robust, and they are the saddle point solutions of the games.

In section 2, we saw that the minimum error expressions (7) and (12) depend only on the absolutely continuous part of the spectral distribution matrix F(ω); namely, the spectral density f(ω). The spectral singularities contribute nothing to the minimum. However, under the present formalization it is not, in general, true that the operators $g_{RT}^e(\omega)$ will belong to the corresponding intersection classes S_{RT} (T = L, Q, R = p, i), if we allow spectra with singularities, in our classes $\mathcal{F}_L$, $\mathcal{F}_Q$. As we will see in sections 4 and 5, this is due to the fact that the operators $g_{RT}^e(\omega)$ are defined a.e. (dω) in $[-\pi,\pi]$, and they do not necessarily have to be limit points of the corresponding sets of trigonometric polynomials, with respect to norms $L_2(d\,F(\omega))$, for all the $F(\omega)$'s in either $\mathcal{F}_L$ or $\mathcal{F}_Q$, that contain singularities. For this reason, and throughout the rest of the paper, we will restrict attention to processes with absolutely continuous spectra. In particular, we will assume that the nominal spectral distribution matrix $F_o(\omega)$, in the definition of the class $\mathcal{F}_L$, is absolutely continuous with density $f_o(\omega)$, and that $H(\omega)$ runs over the absolutely continuous spectra with density denoted by $h(\omega)$. Similarly, the class $\mathcal{F}_Q$ will consist of all $F(\omega)$ that, in addition to the constraints imposed in definition (b), will be absolutely continuous as well. Under those assumptions, the two classes $\mathcal{F}_L$ and $\mathcal{F}_Q$ reduce to classes of spectral density matrices. For notational simplicity, we will frequently omit arguments of functions. In section 4, below, we solve the prediction games on $\mathcal{F}_L xS_{pL}$ and $\mathcal{F}_Q xS_{pQ}$. In section 5, we solve the interpolation games on $\mathcal{F}_L xS_{iL}$ and $\mathcal{F}_Q xS_{iQ}$. In all cases, we state the solutions, and we prove them directly by construction.

4. <u>The Solution of the Prediction Games on $\mathcal{F}_L xS_{pL}$ and $\mathcal{F}_Q xS_{pQ}$</u>

Let $\{\lambda_i^o(\omega),\ \underline{x}_i^o(\omega);\ i=1,\dots n\}$ be the ordered eigenvalues of $f_o(\omega)$ $(\lambda_i^o(\omega)\geq\lambda_{i+1}^o(\omega)$, $i=1,\dots n-1, \forall\omega\varepsilon[-\pi,\pi])$, and the corresponding normalized eigenvectors. Since $(2\pi)^{-1}\,\mathrm{tr}\int_{-\pi}^{\pi}(1-\varepsilon)f_o(\omega)d\omega = (2\pi)^{-1}\int_{-\pi}^{\pi}\sum_{i=1}^{n}(1-\varepsilon)\lambda_i^o(\omega)d\omega = (1-\varepsilon)W<W$, there exists a positive number c such that,

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\sum_{i=1}^{n}\max((1-\varepsilon)\lambda_i^o(\omega),\ c)d\omega = W \tag{16}$$

Let us define the set of functions,

$$\lambda_i^e(\omega) = \max((1-\varepsilon)\lambda_i^o(\omega),\ c),\ i=1,\dots n \tag{17}$$

and the matrix

$$f_L^e(\omega) = \sum_{i=1}^{n}\lambda_i^e(\omega)\ \underline{x}_i^o(\omega)\,(\underline{x}_i^o(\omega))^{*T} \tag{18}$$

$f_L^e(\omega)$ is Hermitian and positive definite, for all $\omega\varepsilon[-\pi,\pi]$, since its smallest eigenvalue is uniformly larger than $c > 0$. Furthermore, $f_L^e(\omega)\varepsilon F_L$, since $f_L^e(\omega)-(1-\varepsilon)f_o(\omega)= \sum_{i=1}^{n} (\lambda_i^e(\omega) - (1-\varepsilon)\lambda_i^o(\omega))\ \underline{x}_i^o(\omega)(\underline{x}_i^o(\omega))^{*T}$ is nonnegative definite for all ω, and (16) holds. We also note that $-\infty<n\ell og\ c = \mathrm{tr}\ \ell og(c.I) \leq \mathrm{tr}\ \ell og\ f_L^e(\omega)$ and $(2\pi n)^{-1}\int_{-\pi}^{\pi} \mathrm{tr}\ \ell og\ f_L^e(\omega)d\omega \leq \ell og((2\pi n)^{-1}\int_{-\pi}^{\pi} \mathrm{tr}\ f_L^e(\omega)d\omega) = \ell og\ \frac{W}{n} < \infty$, from which we conclude that $\mathrm{tr}\ \ell og\ f_L^e(\omega)$ is integrable. Therefore, $\mathrm{tr}\ \ell og\ (f_L^e(\omega))^{-1}$ is also integrable. Finally, since $0 < (f_L^e(\omega))^{-1} \leq c^{-1} I$, $(f_L^e(\omega))^{-1}$ is also integrable. Let $K_{eL} \triangleq \exp[(2\pi n)^{-1}\int_{-\pi}^{\pi} \mathrm{tr}\ \ell og\ f_L^e(\omega)d\omega]$. We consider the matrix $K_{eL}(f_L^e(\omega))^{-1}$, which is easily recognized to be equal to the right-hand side of equation (8), in section 2, for $f_L^e = f$, and which satisfies the requirements of Theorem 7.13 in Wiener et al (1957, 1958). From that we conclude that there exists a factorization of $K_{eL}(f_L^e(\omega))^{-1}$ of the following form.

$$g_{pL}^e(\omega)(g_{pL}^e(\omega))^{*T} = K_{eL}(f_L^e(\omega))^{-1} \quad ; \text{a.e.}(d\omega) \tag{19}$$

where, if $\{A_n^e,\ n\ \varepsilon\ Z\}$ are the Fourier coefficients of $g_{pL}^e(\omega)$, then $A_n^e = 0$ for $n < 0$, and $\det A_o^e = 1$. According to the derivations in Helson et al (1958), $g_{pL}^e(\omega)$ is the element of $\overline{S_p^o}\ (f_L^e(\omega)d\omega)$ that minimizes $e_p(f_L^e(\omega),\ g_{pL}(\omega))$, with respect to $g_{pL}(\omega)\ \varepsilon\ \overline{S_p^o}\ (f_L^e(\omega)d\omega)$. That is,

$$\begin{aligned} &e_p(f_L^e(\omega),\ g_{pL}^e(\omega)) \leq e_p(f_L^e(\omega),\ g_{pL}(\omega)) \\ &;\ \forall\ g_{pL}(\omega)\ \varepsilon\ \overline{S_p^o}\ (f_L^e(\omega)d\omega). \end{aligned} \tag{20}$$

In Lemma 1 below, we prove that $g_{pL}^e(\omega)\ \varepsilon\ S_{pL}$. Theorem 1 establishes that the pair $(f_L^e(\omega),\ g_{pL}^e(\omega))$ is the solution of the prediction game on $F_L x S_{pL}$.

<u>Lemma 1</u> $\quad g_{pL}^e(\omega)\ \varepsilon\ S_{pL}$

<u>Proof</u>

From (19) and $(f_L^e(\omega))^{-1} \leq c^{-1} I$, we conclude that each entry of $g_{pL}^e(\omega)$ is essentially bounded $(d\omega)$. From its Fourier coefficients A_o^e, A_1^e,..., we form the sequence

$\{G_N^e(e^{j\omega})\}$ of the Fejér-Cesaro partial sums,

$$G_N^e(e^{j\omega}) = \frac{1}{N+1} \sum_{k=0}^{N} S_k^e(e^{j\omega}) \tag{21}$$

where

$$S_N^e(e^{j\omega}) = \sum_{i=0}^{N} A_i^e \, e^{j\omega i} \text{ . Evidently, } G_N^e(e^{j\omega}) \in S_p^o.$$

By the usual theory of this sum, each entry of $G_N^e(e^{j\omega})$ converges a.e. ($d\omega$) to the corresponding entry of $g_{pL}^e(\omega)$ boundedly, since $g_{pL}^e(\omega)$ is bounded. Put $h_N(\omega) = G_N^e(e^{j\omega}) - g_{pL}^e(\omega)$. Then, for any $f(\omega) \in F_L$ we have:

$$||h_N(\omega)||^2_{f(\omega)d\omega} = \int_{-\pi}^{\pi} \mathrm{tr}\ h_N(\omega)\ f(\omega)\ h_N^{*T}(\omega)d\omega \leq$$

$$\leq \int_{-\pi}^{\pi} \lambda_{max}(h_N^{*T}(\omega)h_N(\omega))\ \mathrm{tr}\ f(\omega)d\omega.$$

where $\lambda_{max}(\cdot)$ denotes maximum eigenvalue. Since $h_N(\omega) \to 0$ a.e.($d\omega$) boundedly, it is implied that $\lambda_{max}(h_N^{*T}(\omega)\ h_N(\omega)) \to 0$ a.e. ($d\omega$) boundedly. Now, since $\mathrm{tr}\ f(\omega)$ contains no singularities, due to the assumed absolute continuity of the members of F_L, it is concluded that $\lambda_{max}(h_N^{*T}(\omega)\ h_N(\omega)) \to 0$ a.e. ($\mathrm{tr}\ f(\omega)d\omega$). Application of the dominated convergence theorem on $\lambda_{max}(h_N^{*T}(\omega)\ h_N(\omega))$ yields:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \lambda_{max}(h_N^{*T}(\omega)\ h_N(\omega))\ \mathrm{tr}\ f(\omega)d\omega =$$

$$= \int_{-\pi}^{\pi} \lim_{N\to\infty} \lambda_{max}(h_N^{*T}(\omega)\ h_N(\omega))\ \mathrm{tr}\ f(\omega)\ d\omega = 0, \text{ which implies}$$

$$||h_N(\omega)||_{f(\omega)d\omega} \to 0$$

The preceding arguments show that there always exists a sequence of elements of S_p^o, which tends to $g_{pL}^e(\omega)$, under any norm $||\cdot||_{f(\omega)d\omega}$, $f(\omega) \in F_L$. Thus

$g^e_{pL}(\omega) \in S_{pL}$

<u>Remark.</u> The basic constituents for the proof of lemma 1 are: 1) The fact that the eigenvalues of $f^e_L(\omega)$ are bounded away from zero, which implies the a.e. ($d\omega$) boundedness of $g^e_{pL}(\omega)$. 2) The absolute continuity of the members of the class $\mathcal{F}_L$, which permits the transition from the a.e. ($d\omega$) to the a.e. (tr $f(\omega)\, d\omega$) convergence. The above requirements are satisfied for all the other games we consider in the sequel.

<u>Theorem 1</u>

The pair $(f^e_L(\omega), g^e_{pL}(\omega))$ is a saddle point solution of the prediction game on $\mathcal{F}_L x S_{pL}$.

<u>Proof</u>

We have to prove:

$$e_p(f_L, g^e_{pL}) \le e_p(f^e_L, g^e_{pL}) \le e_p(f^e_L, g_{pL}) \;;\; \forall\, f_L \in \mathcal{F}_L \;;\; \forall\, g_{pL} \in S_{pL} \tag{22}$$

The right-hand side inequality in (22) follows from (20) and $\overline{S^o_p} \; (f^e_L\, d\omega) \supset S_{pL}$. Also

$$e_p(f^e_L, g^e_{pL}) = n \exp[(2\pi n)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\; \ell og\; f^e_L(\omega) d\omega] = n\, K_{eL} \tag{23}$$

and

$$e_p(f_L, g^e_{pL}) = (2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}[g^e_{pL}(g^e_{pL})^{*T} f_L] d\omega \tag{24}$$

Combining (24) with (19), we get,

$$e_p(f_L, g^e_{pL}) = (2\pi)^{-1} K_{eL} \int_{-\pi}^{\pi} \mathrm{tr}[f_L (f^e_L)^{-1}] d\omega =$$

$$= (2\pi)^{-1} K_{eL} \int_{-\pi}^{\pi} \sum_{i=1}^{n} (\lambda^e_i(\omega))^{-1} (\underline{x}^o_i(\omega))^{*T} f_L(\omega)\, \underline{x}^o_i(\omega) d\omega \tag{25}$$

Put $\mu_i(\omega) = (\underline{x}^o_i(\omega))^{*T} f_L(\omega)\, \underline{x}^o_i(\omega)$. Since $f_L(\omega) \in \mathcal{F}_L$, $f_L(\omega) - (1-\varepsilon)\, f_o(\omega)$ should be nonnegative definite which implies that

$$\mu_i(\omega) \geq (1-\varepsilon)\ \lambda_i^o(\omega)\ ;\ \forall\ \omega,\ i=1,\ldots n \tag{26}$$

Also

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \text{tr}\ f_L(\omega)d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \mu_i(\omega)d\omega = W \tag{27}$$

From (25) and (23) we obtain,

$$e_p(f_L,\ g_{pL}^e) - e_p(f_L^e,\ g_{pL}^e) = K_{eL} \left((2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \frac{\mu_i(\omega)}{\lambda_i^e(\omega)}\ d\omega - n\right) =$$

$$= \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \left[\ \int_{(1-\varepsilon)\lambda_i^o(\omega) \geq c} \frac{\mu_i(\omega)-(1-\varepsilon)\lambda_i^o(\omega)}{(1-\varepsilon)\lambda_i^o(\omega)}\ d\omega + \int_{(1-\varepsilon)\lambda_i^o(\omega)<c} \frac{\mu_i(\omega)-c}{c}\ d\omega\right] \leq$$

$$\leq \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \left[\ \int_{(1-\varepsilon)\lambda_i^o(\omega) \geq c} \frac{\mu_i(\omega)-(1-\varepsilon)\lambda_i^o(\omega)}{c}\ d\omega + \int_{(1-\varepsilon)\lambda_i^o(\omega)<c} \frac{\mu_i(\omega)-c}{c}\ d\omega\right] =$$

$$= \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \int_{-\pi}^{\pi} \frac{\mu_i(\omega)-\lambda_i^e(\omega)}{c}\ d\omega = \frac{K_{eL}}{2\pi c} \int_{-\pi}^{\pi} \text{tr}(f_L(\omega)-f_L^e(\omega))d\omega = 0$$

and the left-hand side inequality in (22) follows.

We now proceed to the solution of the prediction game on $F_Q x S_{pQ}$. We define the spectral density matrix

$$f_Q^e(\omega) = \left(\frac{2\pi}{n} \sum_{i=1}^{k} c_i\ 1_{D_i}(\omega)\ m^{-1}(D_i)\right) \cdot I \tag{28}$$

where $1_{D_i}(\omega)$ is the indicator function of the set D_i, and $m(\cdot)$ is the Lebesque measure in $[-\pi,\pi]$. It can be seen by inspection that $f_Q^e(\omega) \in F_Q$. Since $c_i > 0$, $m(D_i) > 0$, the eigenvalue of $f_Q^e(\omega)$ is bounded away from zero and from infinity, by

$\frac{2\pi}{n} \min_{i=1,\dots k} \{\frac{c_i}{m(D_i)}\}$, $\frac{2\pi}{n} \max_{i=1,\dots k} \{\frac{c_i}{m(D_i)}\}$ respectively. It follows that $(f_Q^e(\omega))^{-1}$ and $\operatorname{tr} \mathcal{L}og\, (f_Q^e(\omega))^{-1}$ both exist and are integrable. We thus conclude that there exists some $g_{pQ}^e(\omega) \in S_p^o\, (f_Q^e(\omega)d\omega)$, such that

$$g_{pQ}^e(g_{pQ}^e)^{*T} = k_{eQ}(f_Q^e)^{-1} \text{ a.e. } d\omega \tag{29}$$

where
$$k_{eQ} = \exp[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} \mathcal{L}og\, f_Q^e\, d\omega].$$

where the Fourier coefficients of g_{pQ}^e, $\{A_i^e,\ i \in Z\}$, vanish for $i < 0$, and where $\det A_o^e = 1$. Exploiting the assumption of the absolute continuity of all the members of F_Q, we can argue exactly, as in Lemma 1, and establish that $g_{pQ}^e \in S_{pQ} = \bigcap_{f \in F_Q} S_p^o(f\, d\omega)$. Also, g_{pQ}^e is the element of $\overline{S_p^o}\, (f_Q^e\, d\omega)$ which minimizes $e_p(f_Q^e,\, g_{pQ})$. We conclude this section with the following theorem.

<u>Theorem 2</u>

The pair $(f_Q^e,\, g_{pQ}^e)$ given by (28), (29) is a saddle point solution of the prediction game on $F_Q x S_{pQ}$.

<u>Proof</u>. We have to prove that, $e_p(f_Q,\, g_{pQ}^e) \le e_p(f_Q^e,\, g_{pQ}^e) \le e_p(f_Q^e,\, g_{pQ})$; $\forall\, f_Q \in F_Q$; $\forall\, g_{pQ} \in S_{pQ}$. The right-hand side inequality follows from the fact that $S_{pQ} \subset \overline{S_p^o}\, (f_Q^e\, d\omega)$, and that g_{pQ}^e is the minimizing element of $e_p(f_Q^e,\, g_{pQ})$ for $g_{pQ} \in \overline{S_p^o}\, (f_Q^e\, d\omega)$. We thus have:

$$e_p(f_Q,\, g_{pQ}^e) = \frac{1}{2\pi} \operatorname{tr} \int_{-\pi}^{\pi} g_{pQ}^e\, f_Q (g_{pQ}^e)^{*T}\, d\omega =$$

$$= \frac{k_{eQ}}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}[f_Q (f_Q^e)^{-1}]\, d\omega = \frac{k_{eQ}}{2\pi} \sum_{i=1}^{n} \frac{nm(D_i)}{2\pi\, c_i} \int_{D_i} \operatorname{tr} f_Q\, d\omega =$$

$$= n\, k_{eQ} = e_p(f_Q^e,\, g_{pQ}^e)$$

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5. The Solution of the Interpolation Games on $F_L x S_{iL}$ and $F_Q x S_{iQ}$

We start with a proposition:

Proposition 1

If $p(\omega)$ is any nonnegative integrable function in $[-\pi,\pi]$, then the function

$$T(\gamma) = \gamma \int_{-\pi}^{\pi} [\max(\gamma,\ p(\omega))]^{-1}\, d\omega$$

defined for $0 < \gamma \leq \operatorname{ess\,sup}_{\omega} p(\omega)$ is monotonic and continuous.

Proof

For $0 < \gamma_2 < \gamma_1 \leq \operatorname{ess\,sup}_{\omega} p(\omega)$ we have:

$$T(\gamma_1)-T(\gamma_2) = \int_{\gamma_1 \geq p(\omega)} d\omega + \int_{\gamma_1 < p(\omega)} \frac{\gamma_1}{p(\omega)}\, d\omega - \int_{\gamma_2 \geq p(\omega)} d\omega - \int_{\gamma_2 < p(\omega)} \frac{\gamma_2}{p(\omega)}\, d\omega =$$

$$= \int_{\gamma_1 \geq p(\omega) > \gamma_2} d\omega + \int_{\gamma_1 < p(\omega)} \frac{\gamma_1}{p(\omega)}\, d\omega - \int_{\gamma_1 < p(\omega)} \frac{\gamma_2}{p(\omega)}\, d\omega - \int_{\gamma_1 \geq p(\omega) > \gamma_2} \frac{\gamma_2}{p(\omega)}\, d\omega =$$

$$= \int_{\gamma_1 < p(\omega)} \frac{\gamma_1-\gamma_2}{p(\omega)}\, d\omega + \int_{\gamma_1 \geq p(\omega) > \gamma_2} \frac{p(\omega)-\gamma_2}{p(\omega)}\, d\omega > 0.$$

Also, $T(\gamma_1)-T(\gamma_2) \leq \dfrac{\gamma_1-\gamma_2}{\gamma_2} \displaystyle\int_{p(\omega)>\gamma_2} d\omega$, from which the continuity follows.

Now, as in section 4, let $\{\lambda_i^o(\omega),\ \underline{x}_i^o(\omega),\ i=1,\ldots n\}$ be the ordered eigenvalues and the corresponding eigenvectors of the nominal spectral density matrix $f_o(\omega)$. For each eigenvalue we define the function,

$$T_i(\gamma) = \gamma \int_{-\pi}^{\pi} [\max(\gamma,\ (1-\varepsilon)\lambda_i^o(\omega))]^{-1}\, d\omega,\ i=1,\ldots n \tag{29}$$

Due to the monotonicity and continuity of $T_i(\gamma)$, for any positive number $c < 2\pi$, there exists a unique γ_i such that $T_i(\gamma_i) = c$, $i=1,\ldots n$. We put

$\gamma_i(c) = T_i^{-1}(c)$. The inverse mapping $T_i^{-1}(c)$ is also monotonic and continuous. Now, since $(2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} (1-\varepsilon)\lambda_i^o(\omega)d\omega = (1-\varepsilon)\, W < W$, there exists a positive number c^*, such that,

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \max(T_i^{-1}(c^*),\ (1-\varepsilon)\lambda_i^o(\omega))\ d\omega = W \tag{30}$$

Before we proceed further, we will make an assumption, concerning the eigenvectors of $f_o(\omega)$, $\{\underline{x}_i^o(\omega),\ i=1,\ldots n\}$. For the purpose of obtaining closed form solutions, we will assume that $\{\underline{x}_i^o(\omega)\}$ are constant, independent of ω, for every $i=1,\ldots n$. We denote them by $\underline{x}_i^o$ omitting their argument. Thus, we consider the class F_L of spectral density matrices, such that the nominal $f_o(\omega)$ has constant eigenvectors. We note that this is different from requiring that all members of F_L have constant eigenvectors.

We define:

$$\lambda_i^e(\omega) = \max(T_i^{-1}(c^*),\ (1-\varepsilon)\lambda_i^o(\omega)) \tag{31}$$

$$f_L^e(\omega) = \sum_{i=1}^{n} \lambda_i^e(\omega)\ \underline{x}_i^o(\underline{x}_i^o)^{*T} \tag{32}$$

The eigenvalues of $f_L^e(\omega)$ in (32) are bounded away from zero, since they are all uniformly larger than or equal to $T_i^{-1}(c^*) > 0$. Also, $f_L^e(\omega) \in F_L$, due to (32), and to the fact that $f^e(\omega) \geq (1-\varepsilon)\ f_o(\omega)$; $\forall\ \omega\in[-\pi,\pi]$. We define

$$g_{iL}^e(\omega) = 2\pi\left(\int_{-\pi}^{\pi} (f_L^e(\omega))^{-1}d\omega\right)^{-1} (f_L^e(\omega))^{-1} \tag{33}$$

which minimizes $||g_i(\omega)||_{f_L^e d\omega}$, $g_i(\omega) \in \overline{S_i}\ (f_L^e d\omega)$. Since $(2\pi)^{-1} \int_{-\pi}^{\pi} g_{iL}^e(\omega)d\omega = I$, the Fejér-Cesaro partial sums of the Fourier series of $g_i^e(\omega)$ are trigonometric polynomials belonging to S_i. Furthermore, since the entries of $g_{iL}^e(\omega)$ are bounded by (33), the sequence of the Fejér-Cesaro sums will converge dominatedly a.e. $(d\omega)$ to

$g^e_{iL}(\omega)$. The absolute continuity of the members of the class F_L, together with the application of the dominated convergence theorem then implies, in a way similar to that in Lemma 1, that the above sequence will converge to $g^e_{iL}(\omega)$, in the norm $||\cdot||_{f(\omega)d\omega}$, for any $f(\omega) \in F_L$. Thus $g^e_{iL}(\omega) \in S_{iL} = \bigcap_{f(\omega)\in F_L} \overline{S_i}\ (f(\omega)d\omega)$. We now state the solution of the game on $F_L x S_{iL}$.

Theorem 3

The pair $(f^e_L(\omega),\ g^e_{iL}(\omega))$ defined by (31), (32), (33) is a saddle point solution of the interpolation game on $F_L x S_{iL}$.

Proof

We restate the theorem, as follows.

$$e_i(f_L(\omega),\ g^e_{iL}(\omega)) \le e_i(f^e_L(\omega),\ g^e_{iL}(\omega)) \le e_i(f^e_L(\omega),\ g_{iL}(\omega))$$

$$;\ \forall\ f_L(\omega) \in F_L,\ \forall\ g_{iL}(\omega) \in S_{iL}$$

The second inequality follows immediately from section 2, and the fact that $S_{iL} \subset \overline{S_i}\ (f^e_L(\omega)\ d\omega)$.

Put $k'_{eL} = \left(\int_{-\pi}^{\pi} (f^e_L(\omega))^{-1}\ d\omega\right)^{-1}$. The following relationships are valid:

$$e_i(f_L(\omega),\ g^e_{iL}(\omega)) - e_i(f^e_L(\omega),\ g^e_{iL}(\omega)) = (2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ g^e_{iL}(\omega)(f_L(\omega)-f^e_L(\omega))(g^e_{iL}(\omega))^{*T} d\omega =$$

$$= 2\pi \int_{-\pi}^{\pi} \mathrm{tr}[k'_{eL}(f^e_L(\omega))^{-1}\ (f_L(\omega)-f^e_L(\omega))]d\omega =$$

$$= 2\pi \sum_{i=1}^{n} \frac{1}{[\int_{-\pi}^{\pi} (\lambda^e_i(\omega))^{-1} d\omega]^2} \int_{-\pi}^{\pi} \frac{\underline{x}_i^{*T}\ f_L(\omega)\ \underline{x}_i - \lambda^e_i(\omega)}{(\lambda^e_i(\omega))^2} d\omega \qquad (34)$$

Since $f_L(\omega) \in F_L \rightarrow f_L(\omega) \ge (1-\varepsilon)\ f_o(\omega) \rightarrow \underline{x}_i^{*T}\ f_L(\omega)\ \underline{x}_i \ge (1-\varepsilon)\ \underline{x}_i^{*T}\ f_o(\omega)\ \underline{x}_i =$ $= (1-\varepsilon)\lambda^o_i(\omega)$, and $(2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ f_L(\omega)d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \underline{x}_i^{*T}\ f(\omega)\ \underline{x}_i\ d\omega = W$.

$g_{iL}^{e}(\omega)$. The absolute continuity of the members of the class F_L, together with the application of the dominated convergence theorem then implies, in a way similar to that in Lemma 1, that the above sequence will converge to $g_{iL}^{e}(\omega)$, in the norm $||\cdot||_{f(\omega)d\omega}$, for any $f(\omega) \varepsilon F_L$. Thus $g_{iL}^{e}(\omega) \varepsilon S_{iL} = \bigcap_{f(\omega)\varepsilon F_L} \overline{S_i}\ (f(\omega)d\omega)$. We now state the solution of the game on $F_L x S_{iL}$.

Theorem 3

The pair $(f_L^e(\omega),\ g_{iL}^e(\omega))$ defined by (31), (32), (33) is a saddle point solution of the interpolation game on $F_L x S_{iL}$.

Proof

We restate the theorem, as follows.

$$e_i(f_L(\omega),\ g_{iL}^e(\omega)) \leq e_i(f_L^e(\omega),\ g_{iL}^e(\omega)) \leq e_i(f_L^e(\omega),\ g_{iL}(\omega))$$

$$;\ \forall\ f_L(\omega)\ \varepsilon\ F_L,\ \forall\ g_{iL}(\omega)\ \varepsilon\ S_{iL}$$

The second inequality follows immediately from section 2, and the fact that $S_{iL} \subset \overline{S_i}\ (f_L^e(\omega)\ d\omega)$.

Put $k'_{eL} = \left(\int_{-\pi}^{\pi} (f_L^e(\omega))^{-1}\ d\omega\right)^{-1}$. The following relationships are valid:

$$e_i(f_L(\omega),\ g_{iL}^e(\omega)) - e_i(f_L^e(\omega),\ g_{iL}^e(\omega)) = (2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ g_{iL}^e(\omega)(f_L(\omega)-f_L^e(\omega))(g_{iL}^e(\omega))^{*T} d\omega =$$

$$= 2\pi \int_{-\pi}^{\pi} \mathrm{tr}[k'_{eL}(f_L^e(\omega))^{-1}\ (f_L(\omega)-f_L^e(\omega))]d\omega =$$

$$= 2\pi \sum_{i=1}^{n} \frac{1}{\left[\int_{-\pi}^{\pi} (\lambda_i^e(\omega))^{-1} d\omega\right]^2} \int_{-\pi}^{\pi} \frac{\underline{x}_i^{*T}\ f_L(\omega)\ \underline{x}_i - \lambda_i^e(\omega)}{(\lambda_i^e(\omega))^2} d\omega \qquad (34)$$

Since $f_L(\omega)\ \varepsilon\ F_L \rightarrow f_L(\omega) \geq (1-\varepsilon)\ f_o(\omega) \rightarrow \underline{x}_i^{*T}\ f_L(\omega)\ \underline{x}_i \geq (1-\varepsilon)\ \underline{x}_i^{*T}\ f_o(\omega)\ \underline{x}_i =$ $= (1-\varepsilon)\lambda_i^o(\omega)$, and $(2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{tr}\ f_L(\omega)d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \underline{x}_i^{*T}\ f(\omega)\ \underline{x}_i\ d\omega = W.$

Put $v_i = \underline{x}_i^{*T} f_L(\omega)\, \underline{x}_i \geq (1-\varepsilon)\lambda_i^o(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$, $i=1,\dots n$. Then, from (34) we get:

$$e_i(f_L(\omega),\ g_{iL}^e(\omega)) - e_i(f_L^e(\omega),\ g_{iL}^e(\omega)) =$$

$$= 2\pi \sum_{i=1}^{n} \frac{1}{[\int_{-\pi}^{\pi} (\lambda_i^e(\omega))^{-1} d\omega]^2} \left[\int_{T_i^{-1}(c^*) > (1-\varepsilon)\lambda_i^o(\omega)} \frac{v_i(\omega) - T_i^{-1}(c^*)}{(T_i^{-1}(c^*))^2} d\omega + \right.$$

$$\left. + \int_{T_i^{-1}(c^*) \leq (1-\varepsilon)\lambda_i^o(\omega)} \frac{v_i(\omega) - (1-\varepsilon)\lambda_i^o(\omega)}{((1-\varepsilon)\lambda_i^o(\omega))^2} d\omega \right] \leq$$

$$\leq 2\pi \sum_{i=1}^{n} \frac{(T_i^{-1}(c^*))^2}{(c^*)^2} \left[\int_{T_i^{-1}(c^*) > (1-\varepsilon)\lambda_i^o(\omega)} \frac{v_i(\omega) - T_i^{-1}(c^*)}{(T_i^{-1}(c^*))^2} d\omega + \right.$$

$$\left. + \int_{T_i^{-1}(c^*) \leq (1-\varepsilon)\lambda_i^o(\omega)} \frac{v_i(\omega) - (1-\varepsilon)\lambda_i^o(\omega)}{(T_i^{-1}(c^*))^2} d\omega \right] =$$

$$= \frac{2\pi}{(c^*)^2} \sum_{i=1}^{n} \int_{-\pi}^{\pi} (v_i(\omega) - \lambda_i^e(\omega))\, d\omega = 0.$$

The proof is now complete.

Finally, we examine the interpolation game on $F_Q x S_{iQ}$. The result here is summarized in a theorem.

<u>Theorem 4</u>

The pair $(f_Q^e(\omega),\ g_{iQ}^e(\omega))$ which is defined by the expressions,

$$f_Q^e(\omega) = \frac{1}{n} \sum_{\ell=1}^{k} 2\pi\, c_\ell\, 1_{D_\ell}(\omega)\, m^{-1}(D_\ell) \cdot I = \lambda_Q^e(\omega) \cdot I$$

$$g_{iQ}^e(\omega) = 2\pi \left(\int_{-\pi}^{\pi} (f_Q^e(\omega))^{-1} d\omega \right)^{-1} (f_Q^e(\omega))^{-1}$$

is a saddle point solution of the interpolation game on $F_Q x S_{iQ}$.

Proof

Since $f_Q^e(\omega)$ is bounded away from zero and from infinity, due to $c_\ell > 0$, $m(D_\ell) > 0$, $\ell = 1,\dots k$, and since we are considering absolutely continuous spectra only, it will follow again that $g_{iQ}^e(\omega) \in S_{iQ}$. The inequality,

$e_i(f_Q^e(\omega),\ g_{iQ}^e(\omega)) \le e_i(f_Q^e(\omega),\ g_{iQ}(\omega))$; $\forall\ g_{iQ}(\omega) \in S_{iQ}$, follows again from section 2 and $S_{iQ} \subset \overline{S_i}\ (f_Q^e(\omega)\ d\omega)$. We also have:

$$e_i(f_Q(\omega),\ g_{iQ}^e(\omega)) = (2\pi)^{-1}\ \mathrm{tr} \int_{-\pi}^{\pi} g_{iQ}^e(\omega)\ f_Q(\omega)\ g_{iQ}^e(\omega)\ d\omega =$$

$$= 2\pi\left(\int_{-\pi}^{\pi} (\lambda_Q^e(\omega))^{-1}\ d\omega\right)^{-2} \sum_{\ell=1}^{k} \int_{D_\ell} (\lambda_Q^e(\omega))^{-2}\ \mathrm{tr}\ f_Q(\omega)\ d\omega =$$

$$\approx 2\pi\left(\int_{-\pi}^{\pi} (\lambda_Q^e(\omega))^{-1}\ d\omega\right)^{-2} \sum_{\ell=1}^{k} \frac{n^2 m^2(D_\ell)}{4\pi^2 c_\ell^2} \int_{D_\ell} \mathrm{tr}\ f_Q(\omega)\ d\omega =$$

$$= 2\pi\left(\int_{-\pi}^{\pi} (\lambda_Q^e(\omega))^{-1}\ d\omega\right)^{-2} \sum_{\ell=1}^{k} \frac{n^2 m^2(D_\ell)}{2\pi\ c_\ell} =$$

$$= 2\pi n\left(\int_{-\pi}^{\pi} (\lambda_Q^e(\omega))^{-1}\ d\omega\right)^{-1} = e(f_Q^e(\omega),\ g_{iQ}^e(\omega))$$

Finally, the result follows from the last string of relationships.

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6. Conclusions, Discussion

In this paper, we considered the prediction and interpolation problems for vector processes with ill-specified statistical structures. We modeled the uncertainty in the statistical description of the processes, by assuming that their spectral density matrices lie within certain classes of such matrices. Then, we formalized the problems as games, whose saddle point solutions were found for two specific classes of multivariate spectral classes. The first such class (F_L) represents a linear contamination of a nominal spectral density, and it includes an energy constraint. The second class (F_Q) is represented by fixed energy on a finite number of prespecified frequency quantiles.

Both the F_L and the F_Q classes were assumed to consist of absolutely continuous spectra only. If these classes are allowed to include singular spectra as well, the saddle point solutions found, cannot be guaranteed to belong to the appropriate spectral classes. There is an exception for the class F_L, where we can allow the nominal spectrum $F_o(\omega)$ (but not $H(\omega)$) to have singularities at a certain set of points. Then, each member of F_L has singularities at exactly the same points as $F_o(\omega)$. The results that we obtained can be readily extended to include this case. However, when $H(\omega)$ is allowed to have singularities, then a solution does not in general exist. For the latter case and for scalar stationary process an approximate solution is given by Hosoya (1978).

All the derived solutions for the prediction and interpolation games correspond to the eigenvalues with the "flattest" possible tails, or equivalently to measures with the most evenly spread energy. For the F_Q class we obtained identical spectral density matrices for both the prediction and the interpolation solutions, which are diagonal, with a single eigenvalue that is piece-wise constant.

The solutions that we obtained for the F_Q class are not unique, and we just selected the simplest possible. The solutions for the F_L class are also nonunique, in general. All such solutions attain, however, the same saddle value of the game.

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